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GENERALIZED GOLDEN NUMBERS FROM EXTENDED FIBONACCI GROWTH SEQUENCES

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The original concept of Fibonacci´s series is extended to allow more realistic physical conditions (i.e. limited lifespans and different male/female ratios) for the growth of a given population (bacteria, animals, grain in metals, etc.). From these extended fibonacci's sequences is possible to define and construct new golden numbers which may be found in a wider range of natural phenomena because of the infinite possibilities of the evolution (increase and decrease population index) parameters. The generalized mathematical rules which govern the behavior are obtained, as well as a master curve for extended "golden means" φ_n, are deduced.

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Back in the year 1202, the italian matemathician Leonardo Fibonacci[1] was intrigued by the way rabbits breed. Indeed, it is known that at the age of one month a female rabbit is fertile so after only two months an extra couple of rabbits will be suitable for reproduction, and so on. By assuming that no rabbit would die and that each month exactly a male and a female are produced, one can ask how many couples will exist after, let us say, a year. This gentleman did not only found the answer (i.e. 233), but discovered the sequence of numbers that bear his name: the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, … which describes the number of couples after each generation of animals. Notice that each number in the series is the summation of the two numbers preceding it. Later, during the XIX century, the french scholar Edouard Lucas[2] studied further this sequence and extended the concept to what he called "Fibonacci´s general series", consisting of all sequences that can be produced by the above simple summation rule. For example, in the original Fibonacci´s series, the two first numbers are 1 and 1, whereas in Lucas´ version are 2 and 1, and so forth.

Fibonacci´s numbers present fascinating properties and applications[3-13] among which one can mention: the possibility of calculating π from them[4], their use to deduce the Pythagoras figures[5,6], the diffraction problem in Optics[7], phyllotaxis[12-13], etc. Nevertheless, perhaps the most fascinating property of these series is that the ration between one member of the sequence and the previous one, converge to $\varphi = 1.618033….$, the so-called "golden mean", so dear to the ancient cultures[7] that used it in architecture, poetry, mathematics, etc.. Moreover, the convergence behavior can be also be extended to negative numbers, where the ratio will be now $1-\varphi$, as graphically depicted in Figure 1, where the i member of the Fibonacci´s series is represented by *Fib(i).*

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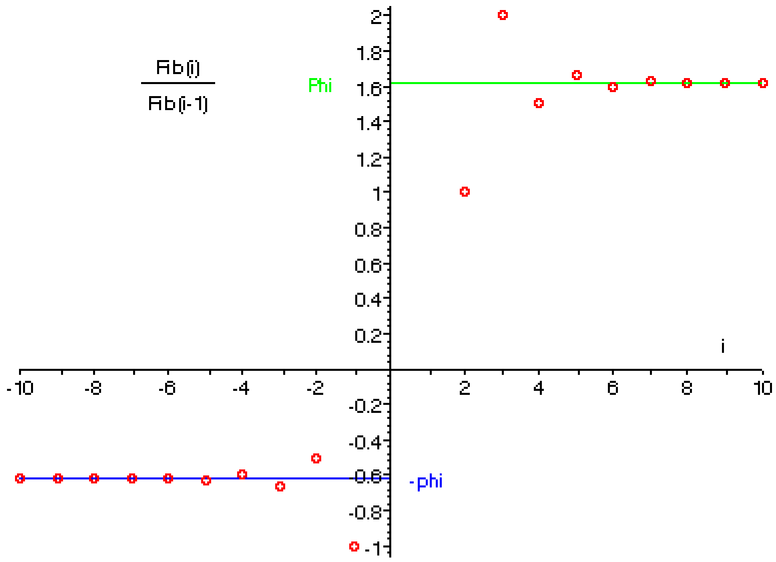

Fig. 1. Convergence of the $\dfrac{Fib(i)}{Fib(i-1)}$ *ratio to* φ *(Phi)*

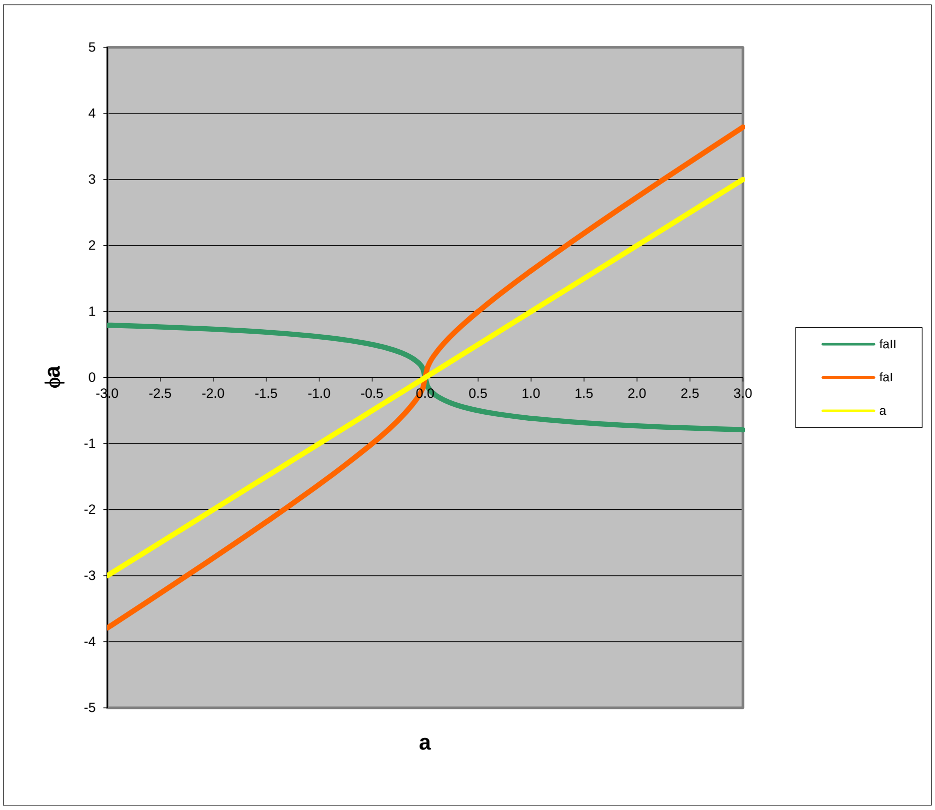

Fig. 2. Master curve of φ_a *for values of e a between -3 and 3.*

In the literature it is often claimed that Fibonacci´s series describe the growth of most biological systems, from larvae to human urban populations[8]. However, as it has been explained above, this series is limited to ideal cases (no deaths and exactly 1 male and 1 female in each generation) with a growth step (ie. number of couples in each genetarion) of 1. In the case of a growth step of 2, the new series is 1, 2, 6, 16, 44, 120, 328, 896,…, where now each number in the sequence is the product of the two previous ones. The ratio now is φ_2 = 2.732050…. Notice that the index describes the step size (therefore, Fibonacci´s original series would be represented by φ_1). For step size 3, the resulting series is 1, 3, 12, 45, 171, 648, 2,457, 9,315…and φ_3 = 3.791287…

This can be generalized to the case of generations consisting of *a* individuals, as summarized in Table 1, where the a negative values imply, obviously, a decrease of the population. The general rules to build the number of individuals present in the i-generation (n_i) are:

$$\text{i>0} \qquad n_{i+1} = |a|\left(n_{i-1} + n_i\right) \tag{1}$$

$$\text{i<0 y } a\text{>0} \qquad n_{i-1} = -1^{|i+1|}|a|\left(n_{i+1} + n_i\right) \tag{2}$$

$$\text{i<0 y } a\text{<0} \qquad n_{i-1} = -1^{|i|}|a|\left(n_{i+1} + n_i\right). \tag{3}$$

$$\lim_{i\to\infty}\frac{n_{i+1}}{n_i} = \varphi_{aI,II} = \frac{a}{|a|}\cdot\frac{|a| \pm \sqrt{|a|\left(|a|+4\right)}}{2} \tag{4}$$

$$n_i = a\frac{|\varphi_{aI}|^i - (-|\varphi_{aI}|)^{-i}}{\sqrt{|a|(|a|+4)}} \tag{5}$$

In equation (4), φ_{aI} represents the ratio of a number and the preceeding one when i > 0, whereas φ_{aII} is the result for i < 0. It must be noticed that the formula is completely general, for it includes irrational and fractional numbers (for example, φ_{aI} = 1 when *a* = 0.5). The ration *a*/|*a*| includes the sign and it can be employed to produce a master curve for φ_a as in Figure 2.

Table 1. First two growth series (a = -2; a = -1)
and two decrease ones (a = 1, standard Fibonacci ; a = 2)

		a			
		a = -2	*a* = -1	*a* = 1	*a* = 2
n_i	n_{-4}	32	3	-3	-32
	n_{-3}	-12	-2	2	12
	n_{-2}	4	1	-1	-4
	n_{-1}	**-2**	**-1**	**1**	**2**
	n_0	**0**	**0**	**0**	**0**
	n_1	**-2**	**-1**	**1**	**2**
	n_2	-4	-1	1	4
	n_3	-12	-2	2	12
	n_4	-32	-3	3	32

Table 2. Values of φ_a provided a is an integer.

a	φ_{aI}	φ_{aII}	$\varphi_{aI} + \varphi_{aII}$
0	0	0	0
1	1,618033988738	-0,618033988750	1
2	2,732050807569	-0,732050807569	2
3	3,791287847478	-0,791287847478	3
4	4,828427124746	-0,828427124746	4
5	5,854101966250	-0,854101966250	5
6	6,872983346207	-0,872983346207	6
7	7,887482193696	-0,887482193696	7
8	8,898979485566	-0,898979485566	8
9	9,908326913196	-0,908326913196	9
10	10,916079783100	-0,916079783100	10

Table 2 contains some specific examples that illustrate the behavior of the series. Indeed, as a grows:

1. φ_{aI} goes to a+1 when i goes to ∞;
2. φ_{aII} goes to –1;
3. $\varphi_{aI} + \varphi_{aII} = a$.

As for equation (5), it allows to obtain the i term of the series. By using a generalization of Binets´s formula [9,10] it is posible to find a given term without a previous knowledge of any other. However, Binet´s formula is only valid when a is an integer. Thus, for the general case of a being a non-integer, equations (1) through (3) must be employed and this possibility enable us to construct extended fibonacci growth sequences.

Notice also that, for a given a, the starting two numbers of the series are arbitrary and the ratio converges to φ_a, as in the original Fibonacci sequence. This is summarized in Table 2.

Since in the generalized series the growth can be either positive or negative, one can relate the Fertility Index, F_I, used in census science, to φ_a-1, when a >0 and $\varphi_{a|}$ +1 when a <0, since they represent the rate of growth ($F_I > 0$) or decrease ($F_I < 0$) of the population, thus allowing to predict population dynamics in many physical systems. In particular, it is interesting to point out that $\varphi_a = 1$ when $a = 0.5$, and $F_I = 0$. Physically, this indicates that the population is not growing. The same situation occurs when $a = -0.5$, $\varphi_a = -1$, so $F_I = 0$. In other words, for $|a| > 0.5$ $F_I > 0$.

Conclusions

By starting from the fundamental mathematical properties of the Fibonacci´s series, it is posible to construct more generalized growth patterns that take into account that: individuals have a limited lifespan, their reprodcutive age is limited, the reproduction rhythms varies among individuals (as, for instance, the numer of offsprings). This possibility enable us to construct extended fibonacci growth sequences and from this fact to define a new set of golden numbers (actually an infinite set) which opens new points of view on our conceptions of armony and beauty. Also, with the above model, only few generations are requiered to have a very good idea of the population dynamics and the probability of catastrophes (i.e. sudden and sharp increases or decreases) can also be evaluated.

References

[1] L. E. Sigler, Fibonacci's Liber Abaci, Springer Verlag (2002).
[2] D. Harkin, On the Mathematical Works of François Edouard Anatole Lucas, Enseignement mathématique **3** (1957).
[2] S. Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Halsted Press (1989).
[4] P. Freguglia, The determination of π in Fibonacci's 'Practica geometriae' in a fifteenth-century manuscript (Italian), in Contributions to the history of mathematics (Modena, 1992).
[5] E. A. Marchisotto, Connections in mathematics: an introduction to Fibonacci via Pythagoras, Fibonacci Quart **31** (1) (1993).
[6] M. Dunton and R. E. Grimm, Fibonacci on Egyptian fractions, Fibonacci Quart **4** (1966).
[7] V. Castaño, The Cornú spiral as a golden mean construction, JJAP **44**, 5009 (2005)
[8] http://evolutionoftruth.com/goldensection/populatn.htm: http://goldennumber.net/
[9] J Kappraff, The relationship between mathematics and mysticism of the golden mean through history, in Fivefold symmetry (River Edge, NJ, 1992).
[10] Binet's Fibonacci Number Formula was derived by Binet in 1843, although the result was known to Euler and to Daniel Bernoulli over a century before
[11] S. Abramovich and G.A. Leonov, Fibonacci numbers revisited: technology-motivated inquiry into a two-parametric difference equation, Int. J. Math. Educ. Sci. & Technol. **39**, 749 (2008)
[12] F.R. Yeatts, A growth-controlled model of the shape of a sunflower head, Math. Biosci. **187**, 205 (2004)
[13] R.V. Jean, A basic theorem on and a fundamental approach to pattern formation on plants, Math. Biosci. **79**, 127 (1986)