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SAMPLED-DATA FORMULATION OF HUMAN SKIN TEMPERATURE DISTRIBUTION PREDICTION DUE TO LASER HEATING

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In this research, we aim to study the boundary heating problem of organisms and to develop a novel methodology for future applications in bioheat transfer and cosmetic laser surgery problems. For biological tissues, the characteristic time needed for accumulating the thermal energy required for propagative transfer to the nearest element within nonhomogeneous inner structures is very large. If the boundary heat source is relatively weak, the Pennes equation is appropriate for describing the heat transfer mechanism. However, for very strong heat source such as laser irradiation, which requires an extremely short time and its heat flux is tremendously high, the thermal wave effect becomes evident during the heat transfer process. In this case, the governing equation is a hyperbolic thermal wave equation. We propose a sampled-data strategy for boundary control of this heat conduction problem modeled by either the Pennes equation or the thermal wave equation. With zero-order-hold, the boundary control law becomes a piecewise constant signal, in which a step change of value occurs at each sampling instant. Through this discretization technique, the governing partial differential equation is dissected into a sequence of constant input problems, to be solved individually for a sampled-data formulation. With this sampled-data formulation, the boundary control problem can be solved and implemented digitally.

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1. Introduction

Laser has been widely used in cosmetic dermatology, which involves skin rejuvenization and the removal of melanosomes, tattoos, hair, etc. [1-5]. In laser treatment, the target molecules or melanosome within the tissue absorb the thermal energy converted from laser light. As the thermal energy is accumulated to a certain level, denaturation or necrosis occurs, leading to damage of structure [6-9]. The goal of treatment is to damage only the target molecule or melanosome without burning the surrounding tissue. Therefore, during the laser heating process, it is important to predict the human skin temperature distribution, as a means to control the extent of thermal damage to the tissue around the treatment target.

For skin temperature and laser heating time control, it is important to obtain the temperature field of the entire treatment region. Since the use of invasive temperature probes is usually not allowed, numerical methods are most widely used for temperature analysis and prediction in biological tissue [10-17]. In this paper, we propose a discrete-time human skin temperature prediction method taking into account of the complex blood perfusion and metabolism, to evaluate accurately the thermal response of the biological tissues, to provide doctors with useful

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data of the thermal analysis of biological tissues, so as to enhance therapeutic effect and patient safety during laser treatment.

The heat conduction equation in materials is based on Fourier's Law. The law serves to define the thermal conductivity of the medium. The effect of heat transfer depends on heat source and thermal conductivity. In biological tissues, the heat source term including the metabolic rate of tissue, the perfusion rate of blood and the volumetric heating etc. Therefore, in 1948, the Pennes bioheat equation is proposed and it is applied to solve the heat transfer of organisms [18].

Due to the Pennes equation is simply and effectively, it is the most commonly used to solve the temperature distribution of skin tissue, such as simulations of hyperthermia, hypothermia and cryosurgery, analysis of thermal diagnostics and thermal comfort, estimation of thermal parameter, and evaluation of burn injury etc. [19-22]. As is well known, the Pennes equation is based on Fourier's law. It described an infinitely fast propagation of thermal signal, obviously incompatible with physical reality. However, its analytical solution cannot satisfy the starting transient temperature response. Thus, the thermal wave equation introduces a relaxation time that is respond to the thermal disturbance. It can be solved the paradox of the Pennes equation [23-27].

The thermal relaxation time for general homogeneous materials ranges from 10^{-14} to 10^{-8} s [7]. Because the heating processes are much longer than this time scale, the phenomenon of thermal wave shows no clear effect in general homogeneous materials. Due to the human skin is non-homogeneous material, it suggests the existence of non-Fourier heat conduction behavior (wave-like), particularly under rapid heating/cooling conditions. In fact, the living tissues are nonhomogeneous material, and accumulating enough energy to transfer to the nearest element would take time. The value of τ in biological bodies has been predicted to be 20-30 s [7]. Mitra *et al* [13] conducted experiments on processed meat and obtained the value of τ is 16 s.

During thermal therapies, the high-energy short-duration heating mode can produce an appropriate and accurate of heat. However, the non-Fourier heat transfer behavior in living tissue plays an important role during rapid heating. The thermal wave effect must be considered. The literature [28] has been shown that the relaxation time (or characteristic time) of tissue will delay the appearance of peak temperature.

The thermal wave model of bioheat transfer [25-26], TWMBT, is the most suitable equation to solve the bioheat transfer problems. However, solving this partial differential equation, PDE, is difficult and complex. At the present, the most common numerical methods are the finite difference approximation method and the boundary element method. The solving mode for these types of heat transfer problems is: First, give a heat source on the skin surface, and then the internal temperature distribution can be predicted. However, to avoid human skin burning down or necrotic, the laser irradiation time must be increased step by step until the longest laser irradiation time is predicted [20, 29-31]. And this prediction time can be ensured the safety of the cosmetic laser surgery. But, this approach has a major drawback, that is, the prediction process of the simulation is time-consuming and tedious.

For the transient problems (For example, heat transfer, wave propagation, vibration, elastic beam, chemical reaction, time delay, etc.), the mathematical models are expressed as to PDEs. Solving these equations is complex and difficult, and they are usually classified as distributed parameter systems, infinite-dimensional systems or finite-dimensional system [32-33]. Therefore, it is very important to establish an effective mode to solve these systems. The sampled-data formulation (finite- dimensional, discrete-time control method) has been applied to solve the complex PDEs widely [34-35]. In this paper, applying zero-order-hold to the control channel leads to a constant control problem within each sampling period. The solution of the distributed system is thus lifted into a sequence of continuous-time signals. The method of separation of variables can then be applied to yield an infinite-dimensional system of ordinary differential equations, referred to as modal equations. In other words, first, the system is segmented in time, and then expanded in frequency. Finally, each modal equation is discretized into a sampled-data formulation. Thus, the boundary control problems have been solved effectively, at the same time, the control and spillover problems have also been avoided in using finite-dimension approximation method [36-38].

In this study, the sampled-data formulation is applied to solve the hyperbolic PDE. It is an innovative, effective and widely applied methodology. In solving the bioheat transfer problems, it can directly control the maximum irradiation temperature of laser on skin tissue, and the time of laser irradiation or the given energy can be derived. This formulation is a reversed heat transfer algorithm that is simply, fast and safe. The results will be applied in cosmetic laser surgery that can estimate the thermal parameters of non-invasive thermal diagnostics, thermal therapy and cryosurgery.

2. Mathematical Analysis

2.1 Mathematical Formulation of Thermal Wave Equation

Consider the human skin, it can be divided into the epidermis, dermis and subcutaneous as Fig. 1. The physical properties of each layer are listed in Table I [16]. Although the three layers in human skin have different physical properties, human skin can be viewed as a single layer based on the following three reasons: First, the thermal properties of these three layers are very similar. Second, the thicknesses of epidermis and dermis relative to subcutaneous are too thin. Third, in references, typical values of physical properties for homogeneous skin tissue have usually been chosen as

$$\rho = 1000 kg/m^3, W_b = 0.5 kg/m^3 \cdot s, W_b = 0.5 kg/m^3 \cdot s, C = C_b = 4200 J/kg \cdot {}^\circ C, \tau = 20 s$$

and the total thickness of human skin is $L = 0.01208 m$ [24,26].

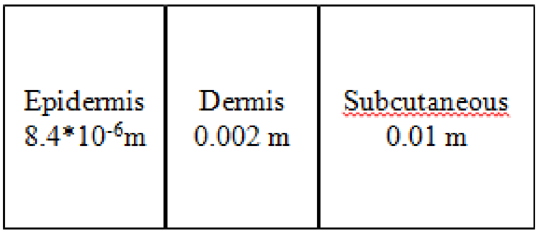

Fig. 1 Schematic diagram of skin structure

Table 1. Physical properties of the human skin

Epidermis	Specific heat $C\left[J/\left(kg \cdot {}^oC\right)\right]$	Volumetric blood perfusion rate $W_b/\rho\,(1/s)$	Thermal conductivity $K\left[W/\left(m \cdot {}^oC\right)\right]$	Thickness $l(m)$	Density $\rho\left(kg/m^3\right)$
Epidermis	3578-3600	0	0.21-0.26	$80*10^{-6}$	1200
Dermis	3200-3400	0.00125	0.37-0.52	0.002	1200
Subcutaneous	2288-3060	0.00125	0.16-0.21	0.01	1000
blood	3770	--	--	--	1060

Pennes bioheat equation (2) [18] is usually applied to solve the bioheat problems, in which the conduction term is based on the well-known Fourier's law [11].

$$q(\vec{r},t) = -K\nabla T(\vec{r},t) \tag{1}$$

where q is heat flux, T is skin temperature, $\vec{r}$ is position vector, K is thermal conductivity

Pennes Bioheat Equation

$$\frac{\partial}{\partial t}T(\vec{r},t) + \frac{W_b C_b}{\rho C}\left[T(\vec{r},t) - T_b\right] = \frac{1}{\rho C}\nabla \cdot \left[K\nabla T(\vec{r},t)\right] + \frac{1}{\rho C}\left[Q_m(\vec{r},t) + Q_r(\vec{r},t)\right] \tag{2}$$

where ρ is density, W_b is the blood perfusion rate, C_b and C are the specific heats of blood and tissue, Q_m is the metabolic rate of tissue (the thermal energy transformed from chemical energy caused by partial metabolism), Q_r is the volumetric heating rate (since heat is incident on the skin, it can be considered as zero), T_b is the temperature of blood, $W_b C_b (T - T_b)$ is the blood flow term (the thermal energy transmitted from in/out controlled volume blood).

However, under the conditions of instantaneous heating and cooling, the conduction term in the heat transfer equation cannot be governed by Fourier's law. Thus, Cattaneo formulated a modified unsteady heat conduction equation (3) [23], and led to the proposition of the thermal wave equation (4) [24-26].

NonFourier’s Law (unsteady heat conduction equation)

$$q(\vec{r},t) + \tau\frac{\partial}{\partial t}q(\vec{r},t) = -K\nabla T(\vec{r},t) \tag{3}$$

where τ is thermal relaxation time in homogeneous substances (range from 10^{-14} to 10^{-8} seconds) or characteristic time in biological systems (range from 20 to 30 seconds).

Thermal Wave Equation

$$\begin{aligned} &\frac{\partial^2}{\partial t^2}T(\vec{r},t) + \frac{\rho C + \tau W_b C_b}{\rho C\tau}\frac{\partial}{\partial t}T(\vec{r},t) + \frac{W_b C_b}{\rho C\tau}\left[T(\vec{r},t) - T_b\right] \\ &= \frac{K}{\rho C\tau}\frac{\partial^2}{\partial x^2}T(\vec{r},t) + \frac{1}{\rho C\tau}\left[Q_m(\vec{r},t) + Q_r(\vec{r},t) + \tau\left(\frac{\partial}{\partial t}Q_m(\vec{r},t) + \frac{\partial}{\partial t}Q_r(\vec{r},t)\right)\right] \end{aligned} \tag{4}$$

The tau value in biological systems is much larger than in homogeneous substances, for the bioheat problems of instantaneous heating and cooling, the simulation results of Pennes equation and thermal wave equation have significantly different. That is to say, the simulation result of Pennes equation is not meet for real situation. Therefore, in this paper, solving these bioheat problems is applied thermal wave equation.

Because of the short time of laser irradiation and the small area for irradiation, the temperature diffusion effect on the skin surface can be neglected. The thermal wave equation can be simplified as a one-dimensional equation.

Assuming thermal conductivity is constant and without metabolic rate and blood perfusion rate, $b_1 = \frac{K}{\rho C \tau}$, $b_2 = \frac{\rho C + \tau W_b C_b}{\rho C \tau}$, $b_3 = \frac{W_b C_b}{\rho C \tau}$, $b_4 = \frac{b_3}{b_1} = \frac{W_b C_b}{K}$, the equation (4) can be deduced as

$$\frac{\partial^2}{\partial t^2} T(x,t) + b_2 \frac{\partial}{\partial t} T(x,t) + b_3 [T(x,t) - T_b] = b_1 \frac{\partial^2}{\partial x^2} T(x,t) \tag{5}$$

For the case of a constant heat flux on the skin surface during a very short laser heating duration (t_s), and assuming the temperature difference is zero in the tissue ($x = L$), the boundary conditions and initial conditions can be described as

$$B.C. \begin{cases} \frac{\partial}{\partial x} T(0,t) = F(t) \\ T(L,t) = T_b \end{cases} \quad F(t) = \begin{cases} -\frac{q}{K} & 0 < t < t_s \\ 0 & t > t_s \end{cases} \quad I.C. \begin{cases} T(x,0) = T_b \\ \frac{\partial}{\partial t} T(x,0) = 0 \end{cases} \tag{6}$$

Introduce a transformation to homogenize the boundary conditions (6):

$$T(x,t) = u(x,t) + F\, v(x) + T_b$$

where $v(x)$ is an auxiliary function whose boundary values satisfy:

$$v''(x) - b_4 v(x) = 0, \quad v'(0) = 1, \text{ and } \quad v(L) = 0$$

then

$$v(x) = \frac{\sinh\left(\sqrt{b_4}(x - L)\right)}{\sqrt{b_4}\cosh\left(\sqrt{b_4} L\right)} \tag{7}$$

Transforming $T(x,t)$ into $u(x,t)$ yields, and eq. (5) and (6) can be deduced as

$$\frac{\partial^2}{\partial t^2} u(x,t) + b_2 \frac{\partial}{\partial t} u(x,t) + b_3 u(x,t) = b_1 \frac{\partial^2}{\partial x^2} u(x,t) \tag{8}$$

$$B.C. \begin{cases} \frac{\partial}{\partial x} u(0,t) = 0 \\ u(L,t) = 0 \end{cases} \quad I.C. \begin{cases} u(x,0) = -F\, v(x) \\ \frac{\partial}{\partial t} u(x,0) = 0 \end{cases} \quad F(t) = \begin{cases} -\frac{q}{K} & 0 < t < t_s \\ 0 & t > t_s \end{cases} \tag{9}$$

First, using separation of variables, let $u(x,t) = X(x)\overline{T}(t)$, and substituting into (8) yields, substituting the boundary conditions (9) into $X(x)$, the $X(x)$ and $\overline{T}(t)$ can be shown as [26,31]

$$X_n = \cos(p_n x)$$

where $\cos(p_n x)$ are eigenvectors, and they are orthogonal to each other.

$$\overline{T}_n(x,t)=\exp\left(-\frac{b_2}{2}t\right)\cdot\begin{cases}\sum_{n=1}^{\infty}\left[A_n\cosh(\omega_n t)+B_n\sinh(\omega_n t)\right], & if\ \left(\frac{b_2}{2}\right)^2>\left(b_3+b_1p_n^2\right)\\ \sum_{n=1}^{\infty}\left[C_n\cos(\gamma_n t)+D_n\sin(\gamma_n t)\right], & if\ \left(\frac{b_2}{2}\right)^2<\left(b_3+b_1p_n^2\right)\end{cases}$$

Secondly, using the superposition principle, the analytical solution of thermal wave equation can be derived as

$$u(x,t)=\exp\left(-\frac{b_2}{2}t\right)\cdot\begin{cases}\sum_{n=1}^{\infty}\left[A_n\cosh(\omega_n t)+B_n\sinh(\omega_n t)\right]\cos(p_n x)\\ \sum_{n=1}^{\infty}\left[C_n\cos(\gamma_n t)+D_n\sin(\gamma_n t)\right]\cos(p_n x)\end{cases}$$

$$\begin{cases}T(x,t)=u(x,t)+Fv(x)+T_b=\\ \exp\left(-\frac{b_2}{2}t\right)\begin{cases}\sum_{n=1}^{\infty}\left[A_n\cosh(\omega_n t)+B_n\sinh(\omega_n t)\right]\cos(p_n x)\\ \sum_{n=1}^{\infty}\left[C_n\cos(\gamma_n t)+D_n\sin(\gamma_n t)\right]\cos(p_n x)\end{cases}+F\frac{\sinh\left(\sqrt{b_4}(x-L)\right)}{\sqrt{b_4}\cosh\left(\sqrt{b_4}L\right)}+T_b\end{cases} \qquad (10)$$

where

$$p_n=\frac{(2n-1)\pi}{2L} \qquad \omega_n=\sqrt{\left(\frac{b_2}{2}\right)^2-\left(b_3+b_1p_n^2\right)} \qquad \gamma_n=\sqrt{\left(b_3+b_1p_n^2\right)-\left(\frac{b_2}{2}\right)^2}$$

$$A_n=\frac{2F}{L\left(b_4+p_n^2\right)} \qquad B_n=\frac{b_2}{2\omega_n}A_n \qquad C_n=\frac{2F}{L\left(b_4+p_n^2\right)} \qquad D_n=\frac{b_2}{2\gamma_n}C_n$$

2.2 Sampled-data formulation

With eq. (8), and boundary condition (9), the general solution for $u(x,t)$ is

$$u(x,t)=\sum_{n=1}^{\infty}q_n(t)\cos(p_n x)$$

then

$$T(x,t)=u(x,t)+Fv(x)+T_b=\sum_{n=1}^{\infty}q_n(t)\cos(p_n x)+Fv(x)+T_b$$

where $q_n(t)$ is the modal displacement for the eigenvector $\cos(p_n x)$ that satisfies the second order differential equation

$$\ddot{q}_n(t)+b_2\dot{q}_n(t)+\left(b_3+b_1p_n^2\right)q_n(t)=0, \qquad n=1,2,... \qquad (11a)$$

The initial condition can be obtained by expanding initial condition function $-Fv(x)$ as infinite series of the eigenfunctions $\cos(p_n x)$

$$q_n(0) = -F\eta_n, \qquad \dot{q}_n(0) = 0, \qquad n = 1,2,\ldots, \tag{11b}$$

where

$$v(x) = \sum_{n=1}^{\infty} \eta_n \cos(p_n x)$$

Applying discrete-time sliding mode control to eq. (8) and (9). Let the boundary control be put through a zero-order-hold

$$F(t) = F_k, \qquad k\tilde{T} \le t < (k+1)\tilde{T}, \qquad k = 0,1,2,\ldots,$$

where $\tilde{T}$ is the sampling period. Define the lifted system

$$\begin{cases} T^k(x,t) \equiv T(x, k\tilde{T} + t), & 0 \le t < \tilde{T} \\ q_i^k(t) \equiv q_i(k\tilde{T} + t), & i = 0,1,2,\ldots \end{cases}$$

The lifting converts a continuous-time signal into an infinite sequence. Each component of the sequence is a function of $t,\ 0 \le t < \tilde{T}$. Since $\frac{\partial^2}{\partial t^2} T(x,t)$ exists for all t, $T(x,t)$ and $\frac{\partial}{\partial t} T(x,t)$ are continuous at each sampling instant $k\tilde{T}$ for all x, thus

$$\begin{cases} T^{k+1}(x,0) = T^k\left(x, \tilde{T}^-\right) \\ \frac{\partial}{\partial t} T^{k+1}(x,0) = \frac{\partial}{\partial t} T^k\left(x, \tilde{T}^-\right) \end{cases}$$

Lifting the modal equation (11) yield

$$\ddot{q}_n^k(t) + b_2 \dot{q}_n^k(t) + \left(b_3 + b_1 p_n^2\right) q_n^k(t) = 0, \quad q_n^k(0) = -F_k \eta_n, \quad \dot{q}_n^k(0) = 0, \quad n = 1,2,\ldots \tag{12}$$

The initial conditions of $q_n^k(t)$ depend on the piecewise constant control F_k, leading to the continuous-time signal $q_i^k(t)$ loses continuity at each sampling instant

$$q_i^{k+1}(0) \ne q_i^k\left(\tilde{T}^-\right)$$

Introduce a new transformation to continue state variable

$$\xi_n^k(t) = q_n^k(t) + F_k \eta_n, \qquad n = 1,2,\ldots$$

Eq. (12) for $\xi_n^k(t)$ becomes

$$\ddot{\xi}_n^k(t)+b_2\dot{\xi}_n^k(t)+\left(b_3+b_1p_n^2\right)\xi_n^k(t)-\left(b_3+b_1p_n^2\right)F\eta_n=0 \quad \xi_n^k(0)=0 \quad \dot{\xi}_n^k(0)=0 \tag{13}$$

The N-mode approximation $T^k(x,t)$ for the kth sampling period is

$$T^k(x,t)=\sum_{n=1}^{N}\xi_n^k(t)\cos(p_nx)+T_b \tag{14}$$

The discrete-time representation for the modal equation (13) can be obtained

$$\xi_n(k+1)=\Phi_n\xi_n(k)+F_k\,\Gamma_n \qquad n=1,2,3\cdots \qquad k=0,1,2\cdots, \tag{15}$$

where $\Phi_n=\begin{bmatrix}0 & 1\\ -\left(b_3+b_1\,p_n^2\right) & -b_2\end{bmatrix}$, $\qquad \Gamma_n=\begin{bmatrix}0\\ \left(b_3+b_1\,p_n^2\right)\eta_n\end{bmatrix}$

The system equation for sampled-data boundary control

$$x((k+1)T)=\Phi x(kT)+\Gamma$$
$$T_N^k(x,kT)=C\,x(kT)+T_b$$

Let $\xi_i^k=\begin{bmatrix}\xi_i^k(0)\\ \dot{\xi}_i^k(0)\end{bmatrix}=x_k=x(kT) \quad \xi_i^{k+1}=\begin{bmatrix}\xi_i^k(T)\\ \dot{\xi}_i^k(T)\end{bmatrix}=\begin{bmatrix}\xi_i^{k+1}(0)\\ \dot{\xi}_i^{k+1}(0)\end{bmatrix}=x_{k+1}=x((k+1)T) \quad i=1,2...$

then

$$\xi_i^{k+1}=\Phi_i\xi_i^k+\Gamma_iF_k \qquad x_{k+1}=\Phi x_k+\Gamma F_k \quad T_N^k(x,0)=\sum_{n=1}^{N}\xi_n^k(0)\cos(p_nx)+T_b=C\,x_k+T_b$$

$$v(x)=\sum_{n=1}^{\infty}\eta_n\cos(p_nx)=\frac{\sinh\left(\sqrt{b_4}(x-L)\right)}{\sqrt{b_4}\cosh\left(\sqrt{b_4}L\right)}$$

$$\eta_n=\frac{\left\langle\dfrac{\sinh\left(\sqrt{b_4}(x-L)\right)}{\sqrt{b_4}\cosh\left(\sqrt{b_4}L\right)}\quad\cos(p_nx)\right\rangle}{\langle\cos(p_nx)\quad\cos(p_nx)\rangle}=\frac{-2}{L\left(b_4+p_n^2\right)}$$

$$x_k=\begin{bmatrix}\xi_0^k=\begin{bmatrix}\xi_0^k(0)\\ \dot{\xi}_0^k(0)\end{bmatrix}\\ \xi_1^k=\begin{bmatrix}\xi_1^k(0)\\ \dot{\xi}_1^k(0)\end{bmatrix}\\ \vdots\\ \xi_N^k=\begin{bmatrix}\xi_N^k(0)\\ \dot{\xi}_N^k(0)\end{bmatrix}\end{bmatrix},\ \left\{\begin{array}{l}C_n\begin{bmatrix}\xi_n^k(0)\\ \dot{\xi}_n^k(0)\end{bmatrix}=\cos p_nx\,\xi_n^k(0), \qquad C_n=[\cos p_nx\ \ 0]\\ C=[C_0\ C_1\ C_2\cdots C_n]\\ C=\{[0\quad 0][\cos p_1x\quad 0][\cos p_2x\quad 0]..[\cos p_Nx\quad 0]\}\\ \Phi=diag\{\Phi_0\ \Phi_1\ \Phi_2\cdots\Phi_N\}\end{array}\right\},\ \Gamma=\begin{bmatrix}\Gamma_0\\ \Gamma_1\\ \Gamma_2\\ \vdots\\ \Gamma_N\end{bmatrix} \tag{16}$$

To employ the system equation for sampled-data boundary control (16), the temperature distribution prediction of human skin can be controlled by adjusting the heat flux of laser. The burn injury problem of human skin is therefore avoided.

3. Results and discussion

To verify the proposed method, we demonstrate the temperature distribution in human skin by laser heating, and the thermal properties for human tissue used by [24-25]

$$C = C_b = 4200 J/kg\cdot°C \text{ 、 } \rho = 1000 kg/m^3 \text{ 、 } W_b = 0.5 kg/m^3\cdot s \text{ 、 } K = 0.2 W/m\cdot°C \text{ 、 }$$
$$\tau = 20s \text{ 、 } Q_r = 0 \text{ 、 } L = 0.01208m \text{ 。}$$

For the central difference approximation of first-order and second-order derivatives:

$$\frac{d}{dx} z_i \approx (z_{i+1} - z_{i-1})/(2h) \qquad \frac{d^2}{dx^2} z_i \approx (z_{i+1} - 2z_i + z_{i-1})/(h^2)$$

For the forward difference and backward difference approximations of first-order derivatives:

$$\frac{d}{dx} z_i \approx (z_{i+1} - z_i)/(h) \qquad \frac{d}{dx} z_i \approx (z_i - z_{i-1})/(h)$$

To use the finite difference approximation, divide the time span of simulation into N_t fine steps and the depth of skin into N_x finite segments.

$$T_{i,j} = T_{i,j}(i\Delta x, j\Delta t)$$

where $\Delta x = 1/N_x$, $\Delta t = T_s/N_t$, $x = i\Delta x$, $t = j\Delta t$, and $v = \Delta t/\Delta x$. The thermal wave equation (5) can be discretized by:

$$T_{i,j+1} = \left[b_1 v^2 \left(T_{i+1,j} + T_{i-1,j}\right) + \left(2 - b_3 \Delta t^2 - 2b_1 v^2\right) T_{i,j} + \left(\frac{b_2}{2}\Delta t - 1\right) T_{i,j-1} \right] \Big/ \left(\frac{b_2}{2}\Delta t + 1\right) \quad (17)$$

The boundary condition and initial condition (6) can be discretized as

$$T_{0,j} = T_{1,j} - F(i\Delta t)\Delta x \qquad T_{Nx,j} = 0 \qquad T_{i,0} = 0 \qquad T_{i,1} = 0 \quad (18)$$

3.1 Comparing the Simulation Results by Different Methods

The solutions of $T(x,t)$ can be calculated by analytical solution (10), sampled-data formulation (16), and finite difference method (18) in MATLAB (simulation software). Fig. 2 shows the temperature distribution prediction of human skin with a constant heat flux ($q = 83.2 kW/m^2$) on skin surface.

Consider the burn injury problems of human skin, the irradiation time of high-energy laser beam on the skin surface cannot be too long. In this study, the heating time is less than 3 seconds. In Fig. 2, although these curves of temperature distribution by three approaches have some differences, the trend of diagram is consistent. There is a larger difference in the initial heating stage, with extending the heating time; the simulation results of three approaches will be gradually close. At the initial heating stage within 1.5 seconds, the tissue temperature has a hysteresis reaction, and then took an instantaneous jump. This phenomenon agrees with the results in References [12, 36].

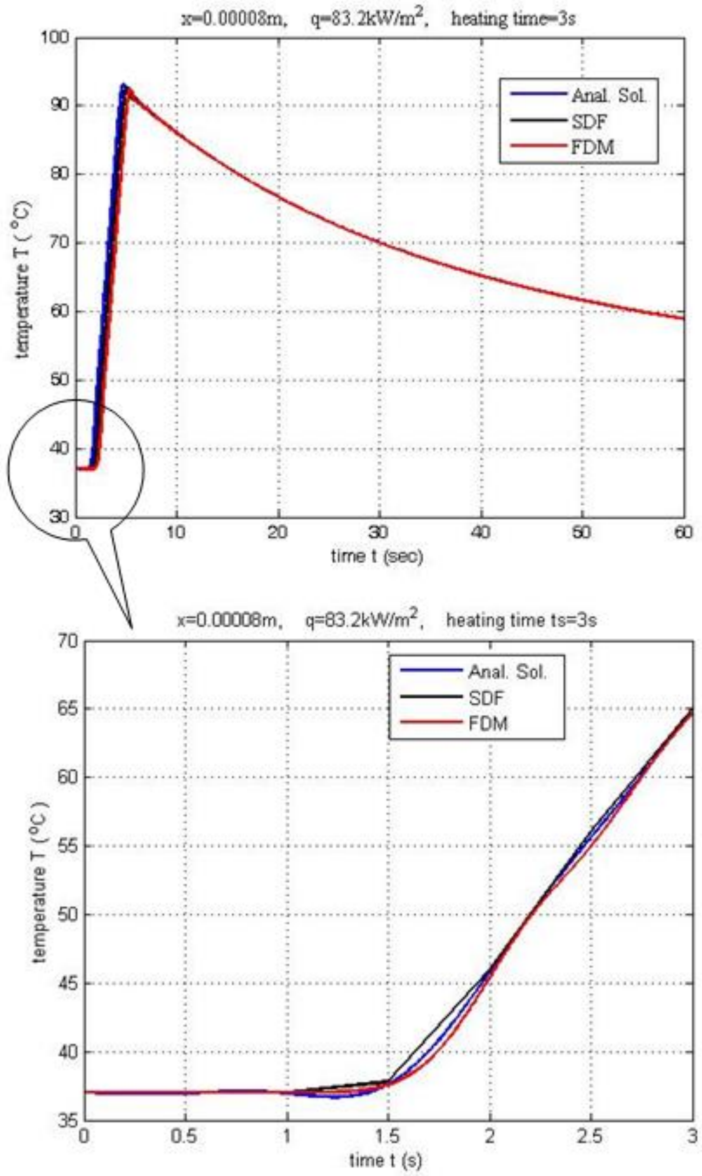

Fig. 2 The comparing figure of skin temperature for a constant heat flux ($q = 83.2kW/m^2$) on skin surface by using Sampled-Data Formulation (SDF), Finite Difference Method (FDM), and Analytical Solution (Anal. Sol.)

Because the thickness of skin is thin, due to software limitations, the Δx cannot too small when the variable x is segmented in space by using finite difference method, otherwise the results have a much larger different (in this paper, the Δx is 10^{-6} m). Similarly, the iteration number n cannot be taken too small in analytical solution, otherwise the correct figures cannot be gotten (in this paper, the $n\ is\ 1000$). In any case, the simulation results of the above two methods will exhibit oscillation pattern. Nevertheless, applying sampled-data formulation method, the oscillation in the figure can be eliminated if the value of N is over 100.

3.2 Comparing the Simulation Results by Different Methods

In the problems of laser heating of human skin, heating duration and characteristic time are the most important physical properties.

Fig. 3 shows the temperature distribution prediction of human skin for a constant heat flux ($q = 83.2kW / m^2$), a constant characteristic time ($\tau = 20\sec$) and different heating durations ($ts = 1, 2, 3\sec$) by using sampled-data formulation. In Figure 3, with extending the heating duration, the temperature of human skin is increase. Consider burn injury problems of human skin, the temperature cannot be over 44oC (Torvi and Dale 1994), thus, the heating duration must be less than 1 second. Controlling the heating duration, it can be avoided human skin burning down or necrotic, that is to say, the burn injury problems of human skin can be therefore solved.

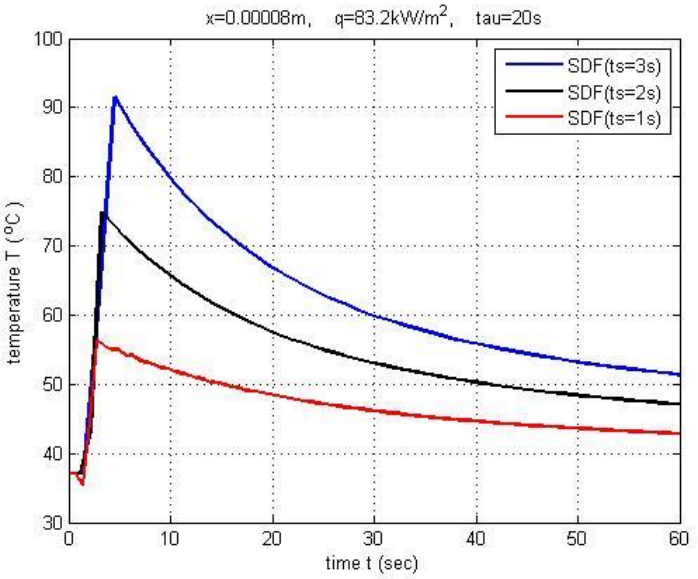

Fig. 3 The comparing figure of skin temperature for different heating times by using Sampled-Data Formulation (SDF)

Fig. 4 shows the temperature distribution prediction of human skin for a constant heating duration($ts = 3\sec$) and different characteristic time ($\tau = 1, 5, 10, 20\sec$) by using the sampled -data formulation. In Figure 4, controlling the characteristic time, it will be demonstrated several phenomena: First, the characteristic time will cause a time delay for heat response, with extending the characteristic time, the time delay is more obvious. Second, the highest skin temperature is decrease with extending the characteristic time. Third, the skin temperature predict have significant difference by different characteristic time in the initial stage. With extending the time, the temperature will be gradually close.

These results indicate the characteristic time is an important parameter under the conditions of instantaneous high-flux heating. When the characteristic time is small or reaches zero, the results of human skin temperature by thermal wave equation will be very close by Pennes equation [31]. (as the blue curve shown in Figure 4)

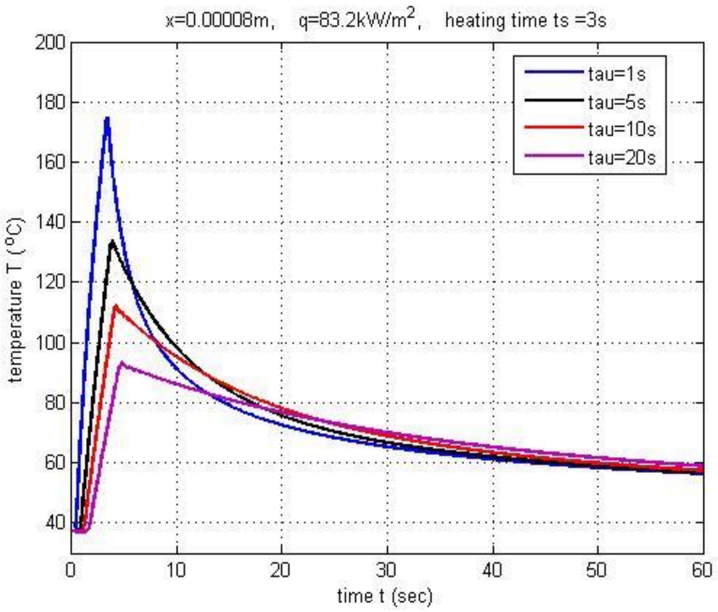

Fig. 4 The comparing figure of skin temperature for different tau values by using Sampled-Data Formulation (SDF)

In summary, the simulation results by using the sampled-data formulation can be calculated more quickly and accurately than by using the finite difference method and the analytical solution. And it has several features:

1. Discrete-time control law: The largest difference of the sampled-data formulation and finite difference method is the design of control laws. After the boundary control problems of PDEs are converted into the sampled-data control mode, many of control design methodology can be used to achieve our expectation. But, the numerical computation of the finite difference method cannot proceed until the boundary condition is set. This approach does not meet the design requirements of the control law.

2. Faster computation: By finite difference method, first it generated isometric mesh first in x-t plane, then calculated the temperature corresponding to each grid point-by-point. In order to increase accuracy, it is required to divide smaller mesh. Thus, increase the amount of computation and the loading of memory will cause the computation time is longer and the calculation speed is slower, even cannot be calculated. By the sampled-data formulation, output or state feedback methods are applied to calculate the temperature distribution in the next time, and only calculating the temperature of target point (x point) , eliminating unnecessary calculations, greatly enhance the computational efficiency of the computer.

4. Conclusions

In this study, the finite-dimensional discrete-time control method of the sampled-data formulation is applied to solve the hyperbolic PDEs. The advantages of this approach: 1. The complex partial differential equations can be solved quickly. 2. The time is segmented and converted into a modal equation. The boundary control problem can be solved effectively. 3. Different physical properties can be selected as the input control term. This approach is applied to solve the engineering problems in view of control. It is an innovative, effective and widely applied methodology. In solving the bioheat transfer problems, it can directly control the maximum irradiation temperature of laser on skin tissue, and the time of laser irradiation or the given energy can be derived. This formulation is a reversed heat transfer algorithm that is simply, fast and safe. The results will be applied in cosmetic laser surgery that can estimate the thermal parameters of non-invasive thermal diagnostics, thermal therapy and cryosurgery.

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